Gameboard Ramsey Numbers

Bode, Jens-Peter
Harborth, Heiko

Veröffentlicht in:
Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft Band 54, 2004, S.63-72

J. Cramer Verlag, Braunschweig

Abhandlungen der BWG · **54**: 63–72 · Braunschweig, Mai 2005

Gameboard Ramsey Numbers*

JENS-P. BODE & HEIKO HARBORTH

Diskrete Mathematik
Technische Universität Braunschweig
38023 Braunschweig, Germany
e-mail: h.harborth@tu-bs.de

Abstract: Sequences $B_n(p,q)$ of connected parts of Euclidean and hyperbolic (p,q)-mosaic graphs are considered. The smallest n such that any 2-coloring of the edges of $B_n(p,q)$ contains a given monochromatic graph G is introduced as gameboard Ramsey number $r_{p,q}(G)$. For $p \geq 4$ it is proved that these Ramsey numbers exist for finitely many graphs only. For $p = 3$ there exist infinitely many numbers $r_{3,q}(G)$. For $p \geq 6$ all gameboard Ramsey numbers are determined.

1. Introduction

The classical Ramsey number $R(G)$ for a given graph G is defined as the minimum number of vertices of a complete graph such that every 2-coloring of its edges contains a monochromatic copy of G. Many generalizations and variations have been considered. However, few investigations are known if other host graphs H_n are used instead of the complete graph K_n. For example, as sequences of host graphs complete bipartite graphs, cube graphs, octahedron graphs, or multipartite graphs are considered in [1–6].

Here as host graphs we choose sequences of gameboards $B_n = B_n(p,q)$ being connected parts of mosaic graphs. A plane graph is called mosaic graph if it is q-regular and all regions are p-gons, $p, q \geq 3$. For given p and q the corresponding mosaic graph originates if starting from a vertex or a p-gon, successively the p-gons and the degrees of vertices are completed. For $(p-2)(q-2) \leq 3$ there are five mosaic graphs, being finite, the five platonic solid graphs. For $(p-2)(q-2) = 4$ and ≥ 5 we have the infinite graphs of the three Euclidean tessellations and of the infinitely many hyperbolic tessellations, respectively. For $(p-2)(q-2) \geq 4$ we define $B_1(p,q)$ to be one p-gon, $B_2(p,q)$ to consist of all p-gons surrounding

* Eingegangen 11.04.2003

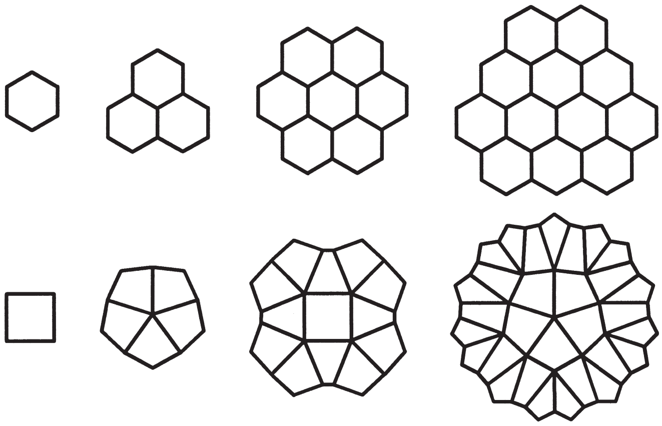

Figure 1: Gameboards $B_n(6,3)$ and $B_n(4,5)$ for $n = 1, \ldots, 4$.

one vertex, and then $B_n(p,q)$ to consist of $B_{n-2}(p,q)$ together with all neighboring p-gons (see Figure 1). Using $B_n(p,q)$ as a sequence of host graphs we define the gameboard Ramsey number $r_{p,q}(G)$ to be the minimum n such that every 2-coloring of the edges of $B_n(p,q)$ contains a monochromatic copy of the given graph G. We always restrict ourselves to connected graphs G having at least two edges since $r_{p,q}(P_2) = 1$ is trivial for the path P_2.

In contrary to the classical Ramsey numbers, the numbers $r_{p,q}(G)$ do not exist in general, not even for all subgraphs of $B_n(p,q)$. In fact, we will see that $r_{p,q}(G)$ exists for infinitely many graphs G if and only if $p = 3$.

2. Euclidean gameboards

For the classical chessboards and for hexagon boards the Ramsey numbers $r_{4,4}(G)$ and $r_{6,3}(G)$ do exist only for two graphs.

Theorem 1. All existing Ramsey numbers for $p = q = 4$ are $r_{4,4}(P_3) = 2$ and $r_{4,4}(P_4) = 3$.

Proof. The only monochromatic subgraphs of the 2-colorings in Figure 2 are paths P_s and subgraphs of cycles C_4. Thus the existence of $r_{4,4}(G)$ is possible for P_3 and P_4 only. The exact values are straightforward (see also Theorems 6 and 7). □

Theorem 2. All existing Ramsey numbers for $p = 6$ and $q = 3$ are $r_{6,3}(P_3) = 2$ and $r_{6,3}(P_4) = 3$.

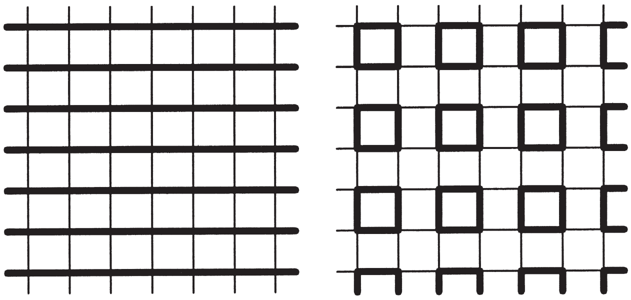

Figure 2: 2-colorings of $B_n(4,4)$.

Proof. The monochromatic components of the 2-coloring in Figure 3 are P_4s. The exact values of $r_{6,3}(P_3)$ and $r_{6,3}(P_4)$ are straightforward (see also Theorems 6 and 7). □

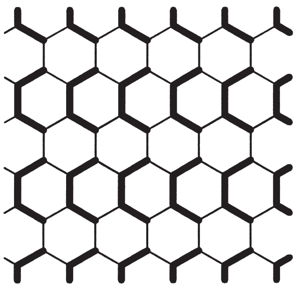

Figure 3: 2-coloring of $B_n(6,3)$.

For triangle boards there are infinitely many graphs G for which $r_{3,6}(G)$ exists. For example the existence of $r_{3,6}(P_n)$ follows from Theorem 13.

Theorem 3. The Ramsey number $r_{3,6}(G)$ exists at most for subtrees of the graph D in Figure 4.

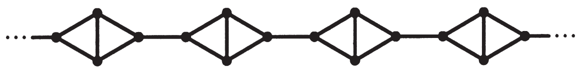

Figure 4: The graph D.

Proof. The monochromatic components in the first 2-coloring in Figure 5 are isomorphic to D in Figure 4. The remaining colorings in Figure 5 do not contain a monochromatic C_3 and C_4, respectively, which are the only cycles in D. □

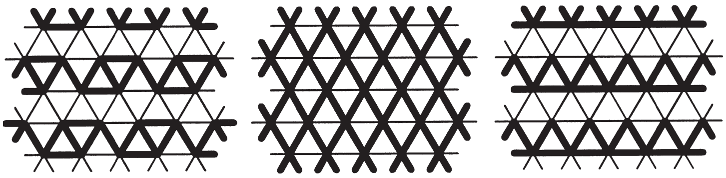

Figure 5: 2-colorings of $B_n(3,6)$.

3. General results

A caterpillar is a tree leaving a path after the removal of all vertices of degree one.

Theorem 4. The Ramsey number $r_{p,q}(G)$ exists only if G is a caterpillar.

Proof. All edges of the plane (p,q)-mosaic graph are 2-colored as follows. Starting with B_2 the border edges of B_i and all new edges of B_{i+2} incident to the border vertices of B_i are colored green for $i \equiv 2 (\mathrm{mod}\ 4)$ and red for $i \equiv 0 (\mathrm{mod}\ 4)$. Then any B_n not covering the initial B_2 of this coloring contains monochromatic caterpillars only, if $p \geq 4$. For $p = 3$ there occur monochromatic triangles in addition. However, if all border edges of B_i are colored green and the remaining 'spokes' are red, monochromatic triangles do not occur (see Figure 6) so that only monochromatic caterpillars are possible for $p = 3$, too. □

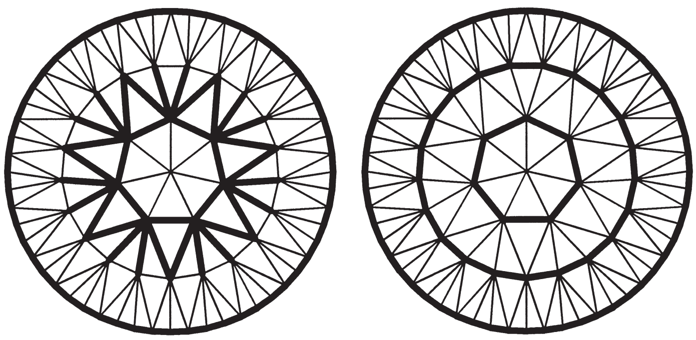

Figure 6: 2-colorings of $B_n(p, q)$.

Theorem 5.

$$r_{p,q}(K_{1,t}) = \begin{cases} \infty & \text{if } t > \lceil q/2 \rceil, \\ 2 & \text{if } t \leq \lceil q/2 \rceil, p \equiv 0 (\text{mod } 2), \\ 2 & \text{if } 2 < t \leq \lceil q/2 \rceil, p \equiv 0 (\text{mod } 2), \\ 1 & \text{if } t = 2, p \equiv 1 (\text{mod } 2). \end{cases}$$

Proof. For $t > \lceil q/2 \rceil$ we construct a 2-coloring of the (p,q)-mosaic graph not containing a monochromatic $K_{1,t}$. Starting with B_2, having its spokes colored appropriately, we color the border edges of a B_i for $i \equiv 0(\text{mod } 2)$ alternatingly in green and red. Beginning at a border vertex incident to an already colored spoke of B_i, the first border edge is colored in the other color. This guarantees that now the spokes from B_i to B_{i+2} can be colored such that no monochromatic $K_{1,t}$ occurs at the border vertices of B_i (note that $q \geq 6$ for $p = 3$).

If $t = 2$ and $p \equiv 1(\text{mod } 2)$ then any 2-coloring of B_1 being an odd cycle C_p contains a monochromatic $K_{1,2}$. In the remaining two cases a 2-coloring of B_1 without a monochromatic $K_{1,t}$ exists proving $r_{p,q} \geq 2$. By the pigeon-hole principle at least $q/2$ edges incident to the central vertex of B_2 have the same color proving $r_{p,q}(K_{1,t}) \leq 2$ for $t \leq \lceil q/2 \rceil$. □

We prove the following theorem although it is contained in Theorem 5.

Theorem 6.

$$r_{p,q}(P_3) = \begin{cases} 1 & \text{if } p \equiv 1(\text{mod } 2), \\ 2 & \text{if } p \equiv 0(\text{mod } 2). \end{cases}$$

Proof. The cycle B_1 can be 2-colored without a monochromatic P_3 if and only if p is even. Since the central vertex of B_2 is incident to at least two equally colored edges it follows $r_{p,q} \leq 2$. □

Theorem 7.

$$r_{p,q}(P_4) = \begin{cases} 2 & \text{if } p = 3 \text{ or } p = 4, q \equiv 1(\text{mod } 2), \\ 3 & \text{if } p = 4, q \equiv 0(\text{mod } 2) \text{ or } p = 6 \text{ or } p \equiv 1(\text{mod } 2), p \geq 5, \\ 4 & \text{if } p \equiv 0(\text{mod } 2), p \geq 8. \end{cases}$$

Proof. Any B_1 can be 2-colored avoiding a monochromatic P_4 so that $r_{p,q}(P_4) \geq 2$.

If $p = 3$ then any triangle of B_2 contains a monochromatic P_3 forcing a P_4 in the other color consisting of edges incident to the vertices of this P_3. This proves $r_{3,q} \leq 2$.

If $p = 4$ and $q \equiv 1 (\mathrm{mod}\ 2)$ then in one quadrangle two edges incident to the central vertex of B_2 are say green. At the end vertices of this green P_3 even a red P_5 is forced. This proves $r_{4,q}(P_4) \leq 2$ if $q \equiv 1 (\mathrm{mod}\ 2)$.

For $p = 4$ and $q \equiv 0 (\mathrm{mod}\ 2)$ the edges around the central vertex of B_2 can be colored alternatingly. To avoid a monochromatic P_4, the border edges of B_2 are forced to be green and red paths P_3. Thus $r_{4,q}(P_4) \geq 3$ for $q \equiv 0 (\mathrm{mod}\ 2)$. For $p \geq 5$ all edges incident to the central vertex of B_2 can be colored green. Then red paths P_3 are forced at their end vertices. Since $p \geq 5$, the remaining edges of the p-gons can be colored such that at most one monochromatic P_3 occurs. This proves $r_{p,q}(P_4) \geq 3$ for $p \geq 5$.

For the following arguments we need a lemma.

Lemma 1. If a vertex in a p-gon, $p \geq 4$, is incident to two equally colored edges of this p-gon and the degrees of all other vertices are at least 3 except possibly for two adjacent vertices if $p \equiv 0 (\mathrm{mod}\ 2)$, then a monochromatic P_4 is guaranteed.

Proof of Lemma 1. Starting with the given monochromatic P_3 a series of monochromatic paths P_3 is forced. However, a monochromatic P_4 cannot be avoided at last. □

For $p \equiv 1 (\mathrm{mod}\ 2)$, $p \geq 5$, the central p-gon of a B_3 contains a monochromatic P_3 and then Lemma 1 guarantees a monochromatic P_4. For $p \equiv 0 (\mathrm{mod}\ 2)$ the central p-gon can be 2-colored alternatingly avoiding a monochromatic P_3. To avoid a red P_4 all other edges incident to one vertex of a red edge of the central p-gon have to be green. Then for that p-gon having two of these green edges, one belonging to the central p-gon, Lemma 1 can be used if $p \leq 6$ since only then the number of vertices of degree two is small enough. This completes $r_{p,q}(P_4) \leq 3$ in the asserted cases of Theorem 7.

For $p \equiv 0 (\mathrm{mod}\ 2)$, $p \geq 8$, we color the edges of the central p-gon of a B_3 alternatingly. Then the sets of adjacent spokes around this central p-gon are colored alternatingly. The two edges incident to the end vertices of the spokes have to be colored differently from the color of the spoke. All remaining edges of the border can be colored such that every p-gon contains at most one additional monochromatic P_3. Thus $r_{p,q}(P_4) \geq 4$ in this case.

For $q \equiv 1 (\mathrm{mod}\ 2)$ in a B_4 there exist two equally colored edges incident to the central vertex and belonging to the same p-gon. Thus Lemma 1 can be used. For $q \equiv 0 (\mathrm{mod}\ 2)$ the edges incident to the central vertex of a B_4 can be colored alternatingly. Then the edges of all p-gons containing the central vertex of B_4 have to be colored alternatingly since otherwise Lemma 1 can be used. To avoid a monochromatic P_4, starting from an end vertex of a monochromatic P_3 on the border of the central B_2, the sets of adjacent spokes around this B_2 are forced to be colored alternatingly green and red. This is impossible for $p \equiv 0 (\mathrm{mod}\ 2)$. Thus $r_{p,q}(P_4) \leq 4$ for $p \equiv 0 (\mathrm{mod}\ 2)$. □

Theorem 8. If $G(s,t)$ for $s \leq t$ and $t \geq 3$ is a path P_4 with $s-2$ and $t-2$ additional edges being incident to the two vertices of degree 2 in the P_4, respectively, then for $p \geq 4$

$$r_{p,q}(G(s,t)) = \begin{cases} \infty & \text{if } s \geq 3 \text{ or } t > \lceil q/2 \rceil, \\ 4 & \text{if } s = 2 \text{ and } t \leq \lceil q/2 \rceil. \end{cases}$$

Proof. The nonexistence for t $> \lceil q/2 \rceil$ follows from Theorem 5. For $s \geq 3$ we describe a 2-coloring for $p \geq 4$ without a monochromatic $G(3,3)$. This will prove $r_{p,q}(G(s,t)) \geq r_{p,q}(G(3,3)) = \infty$ for $s \geq 3$. All edges incident to the central vertex of B_2 are colored green. For every B_{2i} , $i \geq 1$, all border edges between one spoke and the clockwise following spoke are colored differently from the first spoke. Then all spokes from B_{2i} to B_{2i+2} incident to border vertices of degree 3 in B_{2i} are colored differently from the adjacent spoke of B_{2i}. All spokes incident to any remaining border vertex of degree 2 in B_{2i} are differently colored from the equally colored adjacent border edges of B_{2i}. We remark that for $p \geq 5$ at least one border vertex of degree 2 exists between two consecutive border vertices of degree 3 and for $p = 4$ consecutive sets of adjacent spokes from B_{2i} to the border of B_{2i+2} are of different color.

For $p \geq 4$ and $t \geq 3$ it follows $r_{p,q}(G(2,t)) \geq 4$ from the 2-coloring of B_3 where the edges of the central p-gon so as the border edges are colored green and all spokes red. To prove $r_{p,q}(G(2,t)) \leq 4$, consider 2-colorings of B_4. Incident to the central vertex there are at least $\lceil q/2 \rceil$ equally colored central edges, say in green. To avoid a monochromatic $G(2,t)$, all edges adjacent to these green edges and incident to the border vertices of the central B_2 have to be red. Then between any two green central edges the border edges of the central B_2 so as the sets of adjacent spokes from B_2 to B_4 have to be colored alternatingly. However, this is impossible since either the number of these border edges or of these sets of adjacent spokes is even. This contradiction proves $r_{p,q}(G(2,t)) \leq 4$. □

4. Ramsey numbers for $p \geq 6$

Together with the results from the preceding paragraph we now prove that for given p and q, $p \geq 6$, the numbers $r_{p,q}(G)$ exist for finitely many graphs G only and these numbers are determined exactly.

Theorem 9. For $p \geq 6$ the Ramsey numbers $r_{p,q}(P_5)$ do not exist.

Proof. Starting with a B_2 we color all its central edges equally. For $i \geq 1$, all edges adjacent to a spoke of B_{2i} and incident to a border vertex of B_{2i} are colored differently from this spoke of B_{2i}. From every pair of colored border edges of B_{2i} we color the clockwise following edge and the two sets of adjacent spokes differently from the pair. Any clockwise following uncolored border edge together with the remaining uncolored set of adjacent spokes being adjacent to this border edge are colored differently from the preceding border edge. Then all edges of the mosaic graph are colored and for $p \geq 7$ no monochromatic P_5 occurs. For $p = 6$ this coloring contains monochromatic P_5s and we use another coloring.

We start again with equally colored edges incident to the central vertex of B_2. Then every second border edge of B_{2i}, $i \geq 1$, together with all adjacent spokes incident to the border vertices of B_{2i+2} are colored alternatingly. There is one of four possibilities such that no spoke of B_{2i+2} is of the same color as the adjacent spoke of B_{2i}. All remaining border edges of B_{2i} can be colored such that they never have the color of an adjacent spoke of B_{2i}. These colorings for $p = 6$ do not contain monochromatic paths P_5. □

Corollary 1. For $p \geq 6$ the gameboard Ramsey numbers $r_{p,q}(G)$ exist only for $G = P_3$, P_4, $K_{1,t}$, and $G(2, t)$, $3 \leq t \leq \lceil q/2 \rceil$.

Proof. The nonexistence for all other graphs follows from Theorems 2, 4, 5, 8, and 9. The exact values are given in Theorems 2, 5, 6, 7, and 8. □

5. Ramsey numbers for $p = 5$

For $p = 5$ and given q there exist finitely many Ramsey numbers only. Some exact values remain open.

Theorem 10. The Ramsey numbers $r_{5,q}(P_6)$ do not exist.

Proof. Starting with equally colored central edges of B_2, for $i \geq 1$ all edges adjacent to a spoke of B_{2i} and incident to a border vertex of B_{2i} are colored differently from this spoke of B_{2i}. For the remaining border edges of B_{2i} together with all adjacent spokes of B_{2i+2} we use the color different from the color of both already equally colored adjacent spokes of B_{2i+2}. The remaining uncolored sets of adjacent spokes of B_{2i+2} are colored differently from the clockwise preceding spoke of B_{2i+2}. This coloring does not contain a monochromatic P_6. □

Corollary 2. The gameboard Ramsey numbers $r_{5,q}(G)$ exist for $G = P_3$, P_4, $K_{1,t}$, and $G(2,t)$, $3 \leq t \leq \lceil q/2 \rceil$, and possibly for P_5 with 0 to $\lceil q/2 \rceil - 2$ additional edges incident to either the third vertex or to each of the second and fourth vertex.

Proof. The nonexistence for all other graphs follows from Theorems 4, 5, 8, and 10. Exact values are given in Theorems 5, 6, 7, and 8. □

We know that $r_{5,4}(P_5) = 5$ and we conjecture that $r_{5,q}(P_5) = 5$ in general.

6. Ramsey numbers for p = 4

Only finitely many Ramsey numbers exist for $p = 4$ and given q, too.

Theorem 11. The Ramsey numbers $r_{4,q}(P_8)$ do not exist.

Proof. For $q = 4$ Theorem 1 can be used. For $q \geq 5$ all spokes of B_{2i} incident to the border vertices of B_{2i} are colored green for $i \equiv 1 (\mathrm{mod}\ 2)$ and red otherwise. The clockwise following first border edges of B_{2i} being adjacent to a spoke of B_{2i} are colored differently from the spokes and all remaining border edges of B_{2i} get the color of the spokes. Then monochromatic paths P_8 are avoided. □

Theorem 12. For $q \geq 5$ we have

$$r_{4,q}(P_5) = 4.$$

Proof. The edges of the central 4-gon of B_3 are colored alternatingly. The adjacent spokes incident to any vertex of the central 4-gon are colored alternatingly. This is possible such that the four pairs of spokes being opposite edges of 4-gons are colored equally, respectively. Then the border edges are colored differently from their adjacent spokes. This coloring proves $r_{4,q}(P_5) > 3$.

Since $q \geq 5$, two equally colored, say green, central edges of B_4 exist, not being edges of the same 4-gon. Incident to one end vertex of this green P_3 all $q - 1$ edges have to be red to avoid a green P_5. To avoid a red P_5 all edges adjacent to all but one of these red edges have to be green determining a green P_5. This proves $r_{4,q}(P_5) \leq 4$. □

Corollary 3. The gameboard Ramsey numbers $r_{4,q}(G)$ exist for $G = P_3, P_4, K_{1,t}$, and $G(2,t)$, $3 \leq t \leq \lceil q/2 \rceil$, for P_5 if $q \geq 5$, and possibly for P_5, P_6, and P_7 with up to $\lceil q/2 \rceil - 2$ additional edges incident to each vertex of a set of independent inner vertices.

Proof. For all other graphs the nonexistence is proved by Theorems 1, 4, 5, 8, and 11. Exact values are given in Theorems 1, 5, 6, 7, 8, and 12. □

7. Ramsey numbers for p = 3

Opposite to the preceding cases, for triangulations ($p = 3$) there exist infinitely many Ramsey numbers.

Theorem 13. The Ramsey numbers $r_{3,q}(P_s)$ do exist for any s.

Proof. We consider a $B_{2i}(3,q)$ with i being sufficiently large. From those monochromatic components surrounding the central vertex of B_{2i} we choose one, say green, having a vertex with a minimum distance to the border of B_{2i}. If this distance is small and B_{2i} is large enough then this component contains a green P_s. Otherwise this component can be surrounded by a red cycle since either a border vertex of this green component can be connected by a red P_3 to the next border vertex or by a red edge to that border vertex following the next one. This determines a surrounding monochromatic component having a smaller distance to the border of B_{2i} than assumed, a contradiction. $\square$

Due to Theorems 4 and 5 Ramsey numbers $r_{3,q}(G)$ can exist only for caterpillars with maximum degree at most $\lceil q/2 \rceil$. However, even the exact values for $r_{3,q}(P_s)$ are unknown to us so far.

References

[1] BEINEKE, L. W. & A. J. SCHWENK (1976): On a bipartite form of the Ramsey problem. Congr. Numer. **15**: 17–22.

[2] BENCZÚR, A., H. HARBORTH & L. K. JØRGENSEN (1996): Cube graph Ramsey numbers. Abh. Braunschweig. Wiss. Ges. **47**: 151–157.

[3] DAY, D. , W. GODDARD, M. A. HENNING & H. C. SWART (2001): Multipartite Ramsey numbers. Ars Combin. **58**: 23–31.

[4] HARARY, F., H. HARBORTH & I. MENGERSEN (1981): Generalized Ramsey theory of graphs. XII. Bipartite Ramsey sets. Glasgow Math. J. **22**: 31–41.

[5] HARBORTH, H. & I. MENGERSEN (2001): Ramsey numbers in octahedron graphs. Discrete Math. **231**: 241–246.

[6] HATTINGH, J. H. & M. A. HENNING (1998): Bipartite Ramsey theory. Utilitas Math. **53**: 255–258.